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THE PRESSURE OF A NARROW RING-SHAPED PUNCH ON AN ELASTIC HALF-SPACE†

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The linear contact problem for a punch, represented in plan by a narrow ring (of variable thickness), the median line of which is a closed smooth contour, is investigated by the method of matched asymptotic expansions. Galin's hypothesis, according to which the pressure distribution in the transverse direction is largely identical with the solution of the corresponding plane problem, is proved. Various forms of writing the integral equation for the leading term of the asymptotic form of the pressure per unit length are given. Explicit solutions of a number of specific problems are obtained. © 1997 Elsevier Science Ltd. All rights reserved.

Galin [1] obtained the limiting value of the bedding coefficient (relating the settlement of an elastic foundation to the mean pressure $P(s)$ per unit length of the beam) for a beam, rectangular in plane, on the assumption (Galin's hypothesis) that the pressure distribution in a transverse direction is largely the same as that obtained by solving the corresponding plane problem. In particular, for a beam whose cross-section has a rectilinear horizontal base

$$p(s,n)=\frac{P(s)}{\pi\sqrt{h^2-n^2}} \tag{0.1}$$

Galin also formulated the problem of the pressure exerted by a punch, which is ring-shaped in plan, on an elastic half-space [2, Section 2, Chapter II, p. 168].

The first result in solving the axisymmetric problem of a plane ring-shaped punch, that is satisfactory from the practical point of view, was obtained by Yegorov [3, 4]. The approximate extremely simple formula he obtained for the contact pressure is identical with the exact solution in the case of a circular punch, while for a sufficiently narrow ring punch (thickness $2h$) it reduces to (0.1) ($P(s)\equiv(2\pi R)^{-1}Q$, where R is the radius of the median line of the ring-shaped region of contact and Q is the force acting on the punch).

Without dwelling on the papers which have been published on the solution of the contact problem for a wide ring-shaped punch, we note that the first solution of the axisymmetric problem for a narrow ring-shaped punch was obtained by Aleksandrov [5]. Various forms have been given for the leading term of the asymptotic form of the contact pressure in the case of a punch with a plane base, among which is Eq. (0.1). Here the force Q acting on the punch is mainly related to its displacement δ_0 by the equation

$$Q=\frac{\pi E}{2(1-\nu^2)}\frac{2\pi R\delta_0}{\ln(R/h)+\ln 2+a_0};\quad a_0=2{,}079 \tag{0.2}$$

The problem of the indentation of a ring-shaped punch into an elastic half-space due to the action of a vertical applied force was considered in [6, 7].

For a narrow ring-shaped punch, close in plan to circular, it was suggested in [5, 8] that Eqs (0.1) and (0.2) should be used as the approximate solution.

A solution of the contact problem was obtained in [9] for a punch with a base in the form of a narrow rectangle. An equation for the pressure per unit length was obtained using Galin's hypothesis. A discussion of this approach, and also another method of solving the problem are given in [10].

The axisymmetric problem of the pressure of two ring-shaped punches on an elastic half-space was considered in [11]. An approach was proposed in [12] to the more general problem of the interaction of punches represented in plan by regions bounded by circles.

Many papers have been published on the problem of calculating the contact pressures under a roller bearing. The complexity of this problem (the engineering side of the problem is discussed in [13, Section 5.6]) is due primarily to the fact that the contact area is not known in advance and has to be found when solving the problem. Nevertheless, the presence of a narrow contact area enables different approximate solutions, suitable for practical applications, to be found.

The method of matched asymptotic expansions was applied to this problem in [14, 15]. Another approach, based on simplifying expansions related to Galin's hypothesis, was realized in [16]. The original method, which leads to

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results similar to those obtained earlier in [14, 15], was developed in [17]. The case when the contact occurs along somewhat extended narrow regions was considered in [18].

It should be noted that some important features of the asymptotic analysis of the singularly perturbed problems considered were omitted in [14, 15, 17, 18], a mathematically rigorous foundation of which was given for the first time in [19, 22].

We know (see, for example, [13, Section 5.6]), that in the neighbourhood of ring-shaped zones, the stress state of the bodies in contact is essentially three-dimensional and must be considered as such in order to obtain correct results. In other words, whereas in the middle part under the prolate punch the region of local perturbations of a semi-infinite solid is described by a "plane" boundary layer (i.e. Galin's hypothesis is true), in the neighbourhood of the ends it is necessary to construct a "three-dimensional" boundary layer, as was done in [21].

No interpretation was given in [14, 15, 17, 18] of the fact that the limiting problem obtained in the final analysis (for an arbitrary right-hand side) cannot be solved for every ratio ε of the characteristic dimensions of the narrow contact zone. This fact compromises the whole asymptotic analysis of the problem: the asymptotic formulae must "operate" for any fairly small $\varepsilon > 0$. Attention was first given to this in [19–22]; a method of eliminating this drawback was proposed there (details can be found in [23, 24] and Section 7).

Despite the voluminous literature on the subject, the problem of calculating the contact pressures under a punch, which, in plan, takes the form of a narrow curvilinear ring, has not found a proper solution. Here we must emphasize that, at the present time, many authors regard the axisymmetric problem for a ring-shaped punch as a "hard nut" for demonstrating the power of complex mathematical methods for finding so-called exact solutions, but clear results in the problem of the off-centre indentation of a narrow ring-shaped punch into a half-space have not so far been obtained. The approach proposed below, on the other hand, is aimed at finding only an approximate solution (but at the same time an asymptotically exact one), which often turns out to be sufficient for practical calculations.

We also mention two "mechanical" publications [25, 26], which the content of the present paper touches on as far as the method of constructing the asymptotic form is concerned.

1. FORMULATION OF THE PROBLEM

Suppose Γ is a simple closed contour of length $2l$ in the $\mathbf{R}^2$ plane. We will introduce a system of local coordinates (s, n) into its neighbourhood, where s is the arc length and $|n|$ is the distance along the normal. We will denote by ε a small positive parameter and describe a narrow ring of variable thickness $2h(\varepsilon; s) = 2\varepsilon H(s)$, the median line of which forms

$$\Gamma(\varepsilon) = \{(x_1, x_2) \in \mathbf{R}^2\colon\ s \in [0, 2l),\ -\varepsilon H(s) < n < \varepsilon H(s)\}$$

Using the Papkovich–Neuber representation, the contact problem of the gradual indentation of a smooth punch with a plane base in the form of the region $\Gamma(\varepsilon)$ into an elastic half-space to a depth

$$\begin{aligned}
&\mu\Delta_x\mathbf{u}(\varepsilon; \mathbf{x}) + (\lambda + \mu)\operatorname{grad}\operatorname{div}\mathbf{u}(\varepsilon; \mathbf{x}) = 0, \quad \mathbf{x} \in \mathbf{R}^3_- = \{\mathbf{x}\colon x_3 < 0\}\\
&\sigma_{31}(\mathbf{u}; \mathbf{x}) = \sigma_{32}(\mathbf{u}; \mathbf{x}) = 0, \quad x_3 = 0, \quad \mathbf{y} = (x_1, x_2) \in \mathbf{R}^2\\
&\sigma_{33}(\mathbf{u}; \mathbf{y}, 0) = 0, \quad \mathbf{y} \in \mathbf{R}^2\setminus\overline{\Gamma(\varepsilon)}\\
&u_3(\varepsilon; \mathbf{y}, 0) = -\delta, \quad \mathbf{y} \in \Gamma(\varepsilon)
\end{aligned} \tag{1.1}$$

$$\mathbf{u}(\varepsilon; \mathbf{x}) = o(1), \quad |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2} \to \infty$$

reduces (see [2, 27]) to the mixed boundary-value problem of the theory of harmonic functions

$$\begin{aligned}
&\Delta_x\varphi(\varepsilon; \mathbf{x}) = 0, \quad \mathbf{x} \in \mathbf{R}^3_-; \quad \partial_3\varphi(\varepsilon; \mathbf{y}, 0) = 0, \quad \mathbf{y} \in \mathbf{R}^2 \setminus \overline{\Gamma(\varepsilon)}\\
&\varphi(\varepsilon; \mathbf{y}, 0) = -\delta, \quad \mathbf{y} \in \Gamma(\varepsilon); \quad \varphi(\varepsilon; \mathbf{x}) = o(1), \quad |\mathbf{x}| \to \infty
\end{aligned} \tag{1.2}$$

Here λ and μ are the Lamé parameters, $\sigma_{ij}(\mathbf{u})$ are the components of the stress tensor corresponding to the vector $\mathbf{u}$ of the displacements of the points of the half-space, and $\partial_3 = \partial/\partial x_3$.

The pressure of the punch on a semi-infinite body is calculated from the formula

$$p(\varepsilon; \mathbf{y}) = -2\mu(\lambda + \mu)(\lambda + 2\mu)^{-1}\partial_3\varphi(\varepsilon; \mathbf{y}, 0), \quad \mathbf{y} \in \Gamma(\varepsilon) \tag{1.3}$$

Notes 1.1. The assumption that $\delta = \text{const}$ is only made to simplify the discussion and is removed in Section 3. Since, by the maximum principle $\partial_3\varphi(\varepsilon; \gamma, 0) < 0$ when $\gamma \in \Gamma(\varepsilon)$, we can assert a priori that the contact pressure corresponding to the solution of problem (1.1) is positive.

1.2. The parameter ε is introduced formally in order to facilitate the description of the asymptotic method applied to the singularly perturbed boundary-value problem (1.2). In the final formulae we will revert to the actual thickness of the narrow punch $2h(s)$.

2. CONSTRUCTION OF THE ASYMPTOTIC FORM OF THE SOLUTION OF PROBLEM (1.2)

Remote from $\Gamma(\varepsilon)$ the function φ is mainly represented in the form of a simple-layer potential, whose density is concentrated on the contour Γ

$$v(\gamma;\mathbf{x}) = -\frac{1}{2\pi}\int_\Gamma \frac{\gamma(t)dt}{\sqrt{(x_1-y_1(t))^2+(x_2-y_2(t))^2+x_3^2}} \tag{2.1}$$

Here $(y_1(t), y_2(t))$ are the coordinates of a point of Γ, dt is an element of the arc length, and γ is a function to be determined. The quantity $2\mu(\lambda + \mu)(\lambda + 2\mu)^{-1}\gamma(s) \equiv P(s)$ has the meaning of the contact pressure, calculated per unit length of the arc of the median line of the ring-shaped region of contact (see (1.3)).

In the neighbourhood of the contact zone, we construct a solution of the boundary-layer type. To do this, we must first write the Laplace operator in local coordinates

$$(1-nk(s))^{-1}\left\{\frac{\partial}{\partial s}(1-nk(s))^{-1}\frac{\partial}{\partial s}+\frac{\partial}{\partial n}(1-nk(s))\frac{\partial}{\partial n}\right\}+\frac{\partial^2}{\partial x_3^2} \tag{2.2}$$

where $k(s)$ is the curvature of the curve Γ at the point s. Second, we introduce into (2.2) the "fast" variables

$$(\eta_1, \eta_2) = \varepsilon^{-1}(n, x_3) \tag{2.3}$$

keeping the scale for the s coordinate along Γ unchanged. Further, we expand the operator (2.2) in a formal series in powers of ε and separate the principal part

$$\Delta_x - \frac{1}{\varepsilon^2}\Delta_\eta + \frac{1}{\varepsilon^2}\sum_{k=1}^{\infty}\varepsilon^k L_k\left(s, \eta_1; \frac{\partial}{\partial s}, \frac{\partial}{\partial \eta_1}\right)$$

Hence, the boundary layer mentioned turns out to be "plane" and mainly satisfies the following equations, in which the s coordinate occurs as a parameter

$$\begin{gathered}\Delta_\eta w(s;\boldsymbol{\eta}) = 0, \quad \boldsymbol{\eta} \in \mathbf{R}_-^2 = \{\boldsymbol{\eta}:\ \eta_2 < 0\} \\ w(s;\eta_1,0) = -\delta, \quad |\eta_1| < H(s); \quad \partial_2 w(s;\eta_1,0) = 0, \quad |\eta_1| > H(s)\end{gathered} \tag{2.4}$$

We emphasize that relations (2.4) do not completely define the function w—there is no condition imposed on the behaviour of $w(\boldsymbol{\eta})$ as $|\boldsymbol{\eta}| = (\eta_1^2 + \eta_2^2)^{1/2} \to \infty$. Since the functions v (at a distance from $\Gamma(\varepsilon)$) and w (close to $\Gamma(\varepsilon)$) must serve as an approximation to the solution of problem (1.2), they must be matched in the intermediate region. In other words, we must first obtain the asymptotic form of the potential (2.1) as $(n^2 + x_3^2)^{1/2} \to 0$ (which, in the limit, degenerates into a divergent integral). Then, using the result obtained, we establish the nature of the behaviour of the solution of problem (2.4) at infinity and construct the boundary layer. Finally, both representations are matched, eliminating the arbitrariness assumed when solving both limiting problems.

3. REGULARIZATION OF INTEGRAL (2.1)

Suppose $y_1 = f_1(s), y_2 = f_2(s)$ is a natural parametrization of the closed curve Γ without self-intersection points (the functions f_1 and f_2 have the required smoothness). To fix our ideas we will assume that when going round Γ in the direction of increasing s coordinate, the region enclosed by Γ remains on the left.

In the three-dimensional neighbourhood of the contour Γ we change to (s, n, x_3) coordinates, the relation of which to the Cartesian coordinates is given by the formulae

$$x_1 = f_1(s) - nf_2'(s), \quad x_2 = f_2(s) + nf_1'(s), \quad x_3 = x_3 \tag{3.1}$$

where the prime denotes differentiation with respect to s, and n is the distance (taking the sign into account) along the inward normal to the arc $\Gamma \subset \mathbf{R}^2$. We will further introduce polar coordinates (r, φ) in planes normal to Γ: $n = r\cos\varphi, x_3 = r\sin\varphi, \varphi \in [-\pi, 0]$.

The square of the distance between a point in the region of Γ with local coordinates (s, n, x_3) and a point on Γ with coordinate t can be written, by virtue of (3.1), as

$$R_r(s,t)^2 = R_0(s,t)^2 + r^2 - 2r\cos\varphi\{f_2'(s)F_1(s,t) - f_1'(s)F_2(s,t)\} \tag{3.2}$$

$$R_0(s,t) = [F_1(s,t)^2 + F_2(s,t)^2]^{1/2}; \quad F_k(s,t) = f_k(s) - f_k(t)$$

Here $R_0(s, t)$ is the distance between two points on the contour Γ with coordinates s and t.

Taking into account the fact that $f_1'(s)^2 + f_2'(s)^2 = 1$ and $f_1''(s)f_1'(s) + f_2''(s)f_2'(s) = 0$, and that the curvature is expressed by the formula $k(s) = f_2''(s)f_1'(s) - f_1''(s)f_2'(s)$, we obtain

$$R_0(s,t)^2 = (s-t)^2[1 + O(k(s)^2|s-t|^2)], \quad t \to s \tag{3.3}$$

$$f_2'(s)F_1(s,t) - f_1'(s)F_2(s,t) = 2^{-1}k(s)(s-t)^2[1+o(1)], \quad t \to s \tag{3.4}$$

Representing the function (2.1) in the form

$$v(\gamma; s, r, \varphi) = -\frac{\gamma(s)}{2\pi}\int_\Gamma \frac{dt}{R_r(s,t)} - \frac{1}{2\pi}\int_\Gamma \frac{\gamma(t)-\gamma(s)}{R_r(s,t)}dt \tag{3.5}$$

we see that only the first integral in (3.5) will be divergent at the points $r = 0$. We will obtain its first two terms of the asymptotic form as $r = (n^2 + x_3^2)^{1/2} \to 0$. We have

$$\int_\Gamma \frac{dt}{R_r(s,t)} = -\int_{s-l}^{s}\frac{d(s-t)}{R_r(s,t)} + \int_s^{s+l}\frac{d(t-s)}{R_r(s,t)} = -\int_{s-l}^{s}\frac{A(s-t)}{R_r(s,t)}dB(s-t) + \int_s^{s+l}\frac{A(t-s)}{R_r(s,t)}dB(t-s) =$$

$$= -\frac{2r}{R_r(s,s)}\ln r + \sum_\pm \frac{A(l)}{R_r(s,s\pm l)}B(l) + \int_{s-l}^{s}B(s-t)d\frac{A(s-t)}{R_r(s,t)} - \int_s^{s+l}B(t-s)d\frac{A(t-s)}{R_r(s,t)}$$

$$A(x) = \sqrt{x^2+r^2}, \quad B(x) = \ln(x + \sqrt{x^2+r^2})$$

Separating the principal part, we take the limit

$$\int_\Gamma \frac{dt}{R_r(s,t)} = -2\ln r + \sum_\pm \frac{l\ln(2l)}{R_0(s,s\pm l)} +$$

$$+\int_{s-l}^{s}\ln[2(s-t)]d\left(\frac{s-t}{R_0(s,t)} - 1\right) - \int_s^{s+l}\ln[2(t-s)]d\left(\frac{t-s}{R_0(s,t)} - 1\right) + o(1) =$$

$$= -2\ln\frac{r}{2l} + \int_{s-l}^{s+l}\left(\frac{|s-t|}{R_0(s,t)} - 1\right)\frac{dt}{|s-t|} + o(1), \quad r \to 0 \tag{3.6}$$

Hence, from (3.3)–(3.6) we obtain

$$v(\gamma; s, r, \varphi) = \frac{\gamma(s)}{\pi}\left(\ln\frac{r}{2l} - J^0(s)\right) - \frac{1}{\pi}(J\gamma)(s) + o(1), \quad r \to 0 \tag{3.7}$$

$$J^0(s) = \frac{1}{2}\int_{s-l}^{s+l}\left(\frac{1}{R_0(s,t)} - \frac{1}{|s-t|}\right)dt, \quad (J\gamma)(s) = \frac{1}{2}\int_\Gamma \frac{\gamma(t)-\gamma(s)}{R_0(s,t)}dt \tag{3.8}$$

Here $2l$ is the length of the contour Γ.

Note 3.1. In order that the previous calculations should hold good, it is sufficient to require that the contour Γ is smooth, while the function γ, for example, satisfies the Lipschitz condition on it.

Finally, the residue in (3.6) can be estimated by the quantity $O(k_m r \ln [k_m r])$, while in (3.7) it can be estimated by the quantity $O(C_\gamma k_m r \ln [k_m r])$, where k_m is the maximum curvature of the contour, while C_γ is the sum of the Lipschitz constant, multiplied by $2l$, and the maximum of the modulus of γ on Γ.

4. AN EQUATION FOR DETERMINING γ

By virtue of (3.7) the method of matched expansions [28–30] requires a logarithmic increase at infinity of the solution of Eqs (2.4) (see the end of Section 2).

It can be shown directly that the function

$$Y(\boldsymbol{\eta}) = \pi^{-1} \ln\left\{H^{-1}\left|\boldsymbol{\eta} + \sqrt{\boldsymbol{\eta}^2 - H^2}\right|\right\}; \quad (\boldsymbol{\eta} = \eta_1 + i\eta_2) \tag{4.1}$$

where it is understood that, under the radical, we have the branch that is holomorphic in the plane cut along the section $[-H, H]$ of the real axis, and which takes pure imaginary values at the cut edges, satisfies the relations

$$\begin{aligned}
&\Delta_\eta Y(\boldsymbol{\eta}) = 0, \ \ \boldsymbol{\eta} \in \mathbf{R}^2_-; \ \ Y(\eta_1, 0+) = 0, \ \ |\eta_1| < H(s) \\
&\partial_2 Y(\eta_1, 0) = 0, \ \ |\eta_1| > H(s) \\
&Y(\boldsymbol{\eta}) = \pi^{-1}[2|\boldsymbol{\eta}|H(s)^{-1}] + O(|\boldsymbol{\eta}|^{-1}), \ \ |\boldsymbol{\eta}| \to \infty
\end{aligned} \tag{4.2}$$

Notes 4.1. In order to obtain (4.1), it is sufficient to recall that the function under the modulus sign conformally maps this plane with the cut onto the exterior of a circle of radius H.

4.2. When the base of the punch is "wavy" in a longitudinal direction (i.e. $\delta = \delta(s)$), all the discussions remain true if contact is made over the whole region $\Gamma(\varepsilon)$. This leads to the additional requirement

$$\gamma(s) > 0, \ s \in [0, 2l) \tag{4.3}$$

which the solution of the integral equation obtained below must satisfy. Otherwise, instead of a contact problem with a fixed contact area $\Gamma(\varepsilon)$ we obtain a more complex problem with a contact zone—a certain subset of $\Gamma(\varepsilon)$, unknown in advance.

4.3. The leading term of the asymptotic form of the solution of the problem of the indentation with a skewness of a smooth ring-shaped punch with a flat base into an elastic half-space is also sought using the above algorithm. This case reduces to the previous one.

Suppose β_1 and β_2 are small angles of rotation of the punch about the x_1 and x_2 axes, respectively. The penultimate boundary condition in (1.1) can then be changed (see, for example, [27]) to the following

$$u_3(\varepsilon; \mathbf{y}, 0) = -\delta - \beta_2 y_1 + \beta_1 y_2, \ \ \mathbf{y} \in \Gamma(\varepsilon)$$

Changing to local coordinates (s, n) and making the change of variables (2.3), the first boundary conditions in (2.4) can be rewritten in the form

$$w(s; \eta_1, 0) = -\delta - \beta_2 f_1(s) + \beta_1 f_2(s), \ \ |\eta_1| < H(s)$$

4.4. In the general case when $\delta = \delta(s; \eta_1)$, we represent the boundary layer in the form

$$w(s; \boldsymbol{\eta}) = w^0(s; \boldsymbol{\eta}) + \gamma(s) Y(s; \boldsymbol{\eta}) \tag{4.4}$$

where w^0 is the solution of Eqs (2.4) bounded at infinity, with right-hand side $\delta(s; \eta_1)$ (such a solution exists and is unique).

The constant in the asymptotic form

$$w^0(s; \boldsymbol{\eta}) = -c^0(s) + O(|\boldsymbol{\eta}|^{-1}), \ \ |\boldsymbol{\eta}| \to \infty \tag{4.5}$$

is calculated using the second Green's formula for the Laplace operator. Taking (4.1) and (4.5) into account we obtain

$$c^0(s) = -\int_{-H(s)}^{H(s)} \delta(s;\eta_1)\partial_2 Y(\eta_1,0)d\eta_1$$

The pressure under the punch must, of course, be negative. The condition

$$\partial_2 w(s;\eta_1,0) \leq 0, \quad -H(s) < \eta_1 < H(s) \tag{4.6}$$

together with (4.3) ensures contact over the whole set $\Gamma(\varepsilon)$.

We will represent the solution of problem (2.4) in the form

$$w(s;\boldsymbol{\eta}) = -\delta + \gamma(s)Y(s;\boldsymbol{\eta}) \tag{4.7}$$

Note that, by virtue of (4.2), the following asymptotic formula holds (see note 4.2)

$$w(s;\boldsymbol{\eta}) = \pi^{-1}\gamma(s)\ln[2|\boldsymbol{\eta}|H(s)^{-1}] - \delta(s) + O(|\boldsymbol{\eta}|^{-1}), \quad |\boldsymbol{\eta}| \to \infty \tag{4.8}$$

We recall that the function v, at a distance from the contact zone, and w, in the region of the local perturbations, must serve as an approximation to the required solution of problem (1.1). In the intermediate zone, where the other asymptotic representation must also "operate", namely, at distances $|\boldsymbol{\eta}|/H(s)$ of the order of $1/\sqrt{\varepsilon}$, or (which is the same thing) when $r/h(s) = O(\sqrt{\varepsilon})$, they must differ only slightly (by quantities that are small compared with unity). To achieve this, we will derive the required relation connecting γ and δ.

In (3.7) we change to fast coordinates (2.3)

$$v(\gamma;s,\varepsilon\boldsymbol{\eta}) = \pi^{-1}\gamma(s)(\ln[\varepsilon|\boldsymbol{\eta}|(2l)^{-1}] - J^0(s)) - \pi^{-1}(J\gamma)(s) + O(\varepsilon|\boldsymbol{\eta}|k_m \ln(\varepsilon|\boldsymbol{\eta}|k_m)), \quad \varepsilon \to 0 \tag{4.9}$$

and compare (4.9) with (4.8). The difference $v(\gamma; s, \varepsilon\boldsymbol{\eta}) - w(s; \boldsymbol{\eta})$ will be a quantity $O(\sqrt{(\varepsilon)}\ln \varepsilon)$ when $|\boldsymbol{\eta}|/H(\varepsilon) = O(1/\sqrt{\varepsilon})$ provided

$$\gamma(s)\{|\ln\varepsilon| + 2\ln 2 + \ln[lH(s)^{-1}] + J^0(s)\} + (J\gamma)(s) = \pi\delta(s) \tag{4.10}$$

We recall that the solution of integral equation (4.10) must satisfy condition (4.3), which essentially expresses the requirement that the ring-shaped punch must fit the surface of the semi-infinite body over the whole contact area.

Using the formula $\partial_2|f(\boldsymbol{\eta})|^{-1}\mathrm{Re}\,\{\overline{f(\boldsymbol{\eta})}\partial_2 f(\boldsymbol{\eta})\}$ in which $\boldsymbol{\eta} = \eta_1 + i\eta_2$, and the bar indicates complex conjugation, we will seek the derivative $\partial_2 Y$ and show that the contact-pressure distribution corresponding to (4.7) is identical with (0.1).

Note 4.5. In the situation described in note 4.4, the right-hand side of (4.10) changes to

$$\pi c^0(s) = \int_{-H(s)}^{H(s)} \frac{\delta(s;\eta_1)}{\sqrt{H(s)^2 - \eta_1^2}} d\eta_1$$

We emphasize that, in addition to condition (4.3), which the solution of the modified equation (4.10) must obey, it is necessary to satisfy condition (4.6) for the function (4.4), established from the known value of γ.

5. OTHER FORMS OF WRITING THE SOLVING EQUATION

Suppose (to fix our ideas) that the unit circle $|\zeta| < 1$ in the plane of the complex variable $\zeta = \rho e^{i\theta}$ is transformed by means of a conformal mapping

$$z = \omega(\zeta); \quad \omega'(\zeta) = d\omega(\zeta)/d\zeta \neq 0, \ |\zeta| < 1$$

into the simply connected region D in the plane of variable $z = y_1 + iy_2$. Then a certain simple closed curve (situated inside D), which as before will be denoted by Γ, corresponds to the circle ρ = const, $\theta \in [0, 2\pi)$. Here it is appropriate to take the polar angle θ as Γ.

In this case, taking into account the fact that $\chi(\tau) = \rho|\omega'(\rho e^{i\tau})|$ is the coefficient of variation of linear dimensions, we obtain, instead of (2.1)

$$v(\gamma;\mathbf{x}) = -\frac{1}{2\pi}\int_\Gamma \frac{\gamma(\tau)\chi(\tau)d\tau}{\sqrt{(x_1-\operatorname{Re}\omega(\rho e^{i\tau}))^2+(x_2-\operatorname{Im}\omega(\rho e^{i\tau}))^2+x_3^2}}. \tag{5.1}$$

The square of the distance between the point with local coordinates (θ, n, x_3) and a point on the contour with coordinate τ, recalling the notation of Section 4, can be written in the form

$$R_r(\theta,\tau)^2 = R_0(\theta,\tau)^2 + r^2 - \\ -r\cos\varphi\left\{\left[\omega(\rho e^{i\theta})-\omega(\rho e^{i\tau})\right]\frac{e^{-i\theta}\overline{\omega'(\rho e^{i\theta})}}{|\omega'(\rho e^{i\theta})|}+\left[\overline{\omega(\rho e^{i\theta})}-\overline{\omega(\rho e^{i\tau})}\right]\frac{e^{i\theta}\omega'(\rho e^{i\theta})}{|\omega'(\rho e^{i\theta})|}\right\} \tag{5.2}$$

Here $R_0(\theta, \tau) = |\omega(\rho e^{i\theta}) - \omega(\rho e^{i\tau})|$ is the distance between two points belonging to Γ.
Since, as $\tau \to 0$, we have

$$R_0(\theta,\tau)^2 = 4\rho^2|\omega'(\rho e^{i\theta})|^2\sin^2[(\theta-\tau)/2]\{1+O(|\omega''(\rho e^{i\theta})/\omega'(\rho e^{i\theta})|(\theta-\tau))\}$$

the following inequality holds

$$\left|\int_\Gamma\frac{\chi(\tau)d\tau}{\sqrt{R_0(\theta,\tau)^2+r^2}}-\int_{\theta-\pi}^{\theta+\pi}\frac{d\tau}{\sqrt{(\theta-\tau)^2+(r/\chi(\theta))^2}}\right|\leqslant \text{const}$$

On the other hand, since the expression in braces in (5.2) is a quantity $O(\chi)(\theta)^2k_m(\theta-\tau)^2$ as $\tau \to \theta$, we have the limit

$$\left|\int_\Gamma\frac{\chi(\tau)d\tau}{R_r(\theta,\tau)}-\int_\Gamma\frac{\chi(\tau)d\tau}{\sqrt{R_0(\theta,\tau)^2+r^2}}\right|\leqslant \text{const}\left|rk_m\ln\frac{r}{\chi(\theta)}\right|,\quad r\to 0$$

$$k_m = \max_{\theta\in[0,2\pi)} k(\theta),\quad k(\theta)=\frac{1}{\chi(\theta)}\left\{1+\operatorname{Re}\frac{\rho e^{i\theta}\omega''(\rho e^{i\theta})}{\omega'(\rho e^{i\theta})}\right\}$$

These expressions enable us to omit complicated calculations and to write the final formula, namely, the asymptotic form of the function (5.1)

$$v(\gamma;\theta,r,\varphi) = \pi^{-1}\gamma(\theta)\{\ln[r/(2\pi\chi(\theta))]-J_1^0(\theta)\}-\pi^{-1}(J_1\gamma)(\theta)+o(1),\quad r\to 0$$

$$J_1^0(\theta)=\frac{1}{2}\int_{\theta-\pi}^{\theta+\pi}\left(\frac{\chi(\tau)}{R_0(\theta,\tau)}-\frac{1}{|\theta-\tau|}\right)d\tau,\quad (J_1\gamma)(\theta)=\frac{1}{2}\int_0^{2\pi}\frac{\gamma(\tau)-\gamma(\theta)}{R_0(\theta,\tau)}\chi(\tau)d\tau$$

We introduce the integral operators

$$(\mathbf{J}\gamma)(\theta)=\frac{1}{2}\int_0^{2\pi}\frac{\gamma(\tau)-\gamma(\theta)}{2|\sin[(\tau-\theta)/2]|}d\tau,\quad (k_1\gamma)(\theta)=\frac{1}{2}\int_{\theta-\pi}^{\theta+\pi}K_1(\theta,\tau)\gamma(\tau)d\tau \tag{5.3}$$

$$K_1(\theta,\tau)=\chi(\tau)R_0(\theta,\tau)^{-1}-2^{-1}|\sin[(\tau-\theta)/2]|^{-1}$$

Taking into account the fact that

$$\frac{1}{2}\int_{\theta-\pi}^{\theta+\pi}\left(\frac{1}{2|\sin[(\tau-\theta)/2]|}-\frac{1}{|\theta-\tau|}\right)d\tau = 2\ln 2-\ln\pi \tag{5.4}$$

we can reduce the equation for the density γ to the following form

$$\gamma(\theta)\{|\ln\varepsilon|+4\ln 2+\ln[\chi(\theta)H(\theta)^{-1}]\}+(\mathbf{J}\gamma)(\theta)+(k_1\gamma)(\theta)=\pi\delta(\theta) \tag{5.5}$$

Example. The Zhukovskii function $R_1^{-1}z = 2^{-1}(\zeta + \zeta^{-1})$ transforms the exterior of the unit circle into a plane with a rectilinear cut. An ellipse with semiaxes $R_1(\rho + \rho^{-1})/2$, $R_1(\rho - \rho^{-1})/2$ and eccentricity $e = 2/(\rho + \rho^{-1})$ corresponds to a circle of radius $\rho > 1$ in the z plane. We have

$$\begin{aligned}
&|\omega'(\rho e^{i\tau})| = 2^{-1} R_1(1+\rho^{-2})(1-e^2\cos^2\tau)^{1/2}\\
&R_0(\theta,\tau) = R_1\rho(1+\rho^{-2})|\sin[(\tau-\theta)/2]|(1-e^2\cos^2[(\theta+\tau)/2])^{1/2}\\
&K_1(\theta,\tau) = {1/2}^{-1}\operatorname{sign}(\theta-\tau)e^2\sin[(\theta+3\tau)/2]\times\\
&\times\{1-e^2\cos^2[(\theta+\tau)/2]+(1-e^2\cos^2[(\theta+\tau)/2])^{1/2}(1-e^2\cos^2\tau)^{1/2}\}^{-1}
\end{aligned}\tag{5.6}$$

We will now assume that the curve Γ is star-like with respect to the origin of coordinates. It can then be parametrized using the polar angle $y_1 = \rho(\theta)\cos\theta$, $y_2 = \rho(\theta)\sin\theta$; here

$$R_0(\theta,\tau)^2 = (\rho(\theta)-\rho(\tau))^2 + 4\rho(\theta)\rho(\tau)\sin^2[(\theta-\tau)/2]$$

We will change the variable in formulae (3.8). Taking into account the fact that the length element ds is equal to $\xi(\tau)d\tau$, where $\xi(\tau) = (\rho'(\tau)^2 + \rho(\tau)^2)^{1/2}$, while the arc length is expressed by an integral, we obtain

$$\begin{aligned}
&(J\gamma)(S(\theta)) = \frac{1}{2}\int_0^{2\pi}\frac{\gamma(\tau)-\gamma(\theta)}{R_0(\theta,\tau)}\xi(\tau)d\tau \equiv (J_3\gamma)(\theta)\\
&J^0(S(\theta)) = J_3^0(\theta) + \ln\xi(\theta) + \ln(\pi/l)\\
&S(\theta) = \int_0^\theta \xi(\tau)d\tau,\quad J_3^0(\theta) = \frac{1}{2}\int_{\theta-\tau}^{\theta+\tau}\left(\frac{\xi(\tau)}{R_0(\theta,\tau)} - \frac{1}{|\theta-\tau|}\right)d\tau
\end{aligned}$$

Equation (4.10) correspondingly becomes

$$\gamma(\theta)\{|\ln\varepsilon| + 2\ln 2 + \ln[\pi(\rho'(\theta)^2+\rho(\theta)^2)^{1/2}H(s)^{-1}] + J_3^0(\theta)\} + (J_3\gamma)(\theta) = \pi\delta(\theta)\tag{5.7}$$

We introduce the integral operator

$$(k_3\gamma)(\theta) = \frac{1}{2}\int_{\theta-\pi}^{\theta+\pi}K_3(\theta,\tau)\gamma(\tau)d\tau,\quad K_3(\theta,\tau) = \frac{\xi(\tau)}{R_0(\theta,\tau)} - \frac{1}{2|\sin[(\theta-\tau)/2]|}$$

Recalling that the operator J is defined by the first formula of (4.3), we can rewrite (5.7) in the following final form

$$\gamma(\theta)\{|\ln\varepsilon| + 4\ln 2 + \ln[\xi(\theta)H(\theta)^{-1}]\} + (\mathbf{J}\gamma)(\theta) + (k_3\gamma)(\theta) = \pi\delta(\theta)\tag{5.8}$$

6. THE PROPERTIES OF THE OPERATOR J AND THE APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION ON Γ

The properties of the operator **J** were fully investigated in [22–24]. In particular, it was shown by direct integration that the following equations hold

$$\begin{aligned}
&(\mathbf{J}\varphi_{\pm k})(\theta) = -\lambda_k\varphi_{\pm k}(\theta)\quad (k = 1,2,\ldots)\\
&\varphi_{\pm k}(\theta) = \cos k\theta \pm i\sin k\theta,\quad \lambda_k = 2\sum_{j=0}^{k-1}\frac{1}{2j+1}
\end{aligned}\tag{6.1}$$

where, by formula (0.132) of [31]

$$0 < \lambda_k - (\ln k + 2\ln 2 + \mathrm{C}) \leq 1/(24k^2)\quad (k = 1,2,\ldots)\tag{6.2}$$

($\mathrm{C} = 0.577$ is Euler's constant).

We will indicate, by a simple example, the difficulties that arise when constructing the solution of Eqs (4.10), (4.11), (5.5), (5.7) and (5.8).

Consider the case of a ring-shaped punch of constant thickness $2h(\varepsilon) = 2\varepsilon R$, when the contour Γ coincides with a circle of radius R. The solving equation then has the form (see Section 5)

$$\gamma(\theta)(|\ln \varepsilon| + 4\ln 2) + (J\gamma)(\theta) = \pi\delta(\theta) \tag{6.3}$$

Suppose the right-hand side in (6.3) is represented by a Fourier series

$$\delta(\theta) = \frac{A_0}{2} + \sum_{k=1}^{\infty}(A_k \cos k\theta + B_k \sin k\theta) \tag{6.4}$$

We shall also attempt to find a solution in the form of a series

$$\gamma(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\theta + b_k \sin k\theta) \tag{6.5}$$

Substituting (6.4) and (6.5) into (6.3) and taking (6.1) into account we obtain

$$a_0 = \frac{\pi A_0}{|\ln \varepsilon| + 4\ln 2}, \quad \begin{Bmatrix} a_k \\ b_k \end{Bmatrix} = \frac{1}{|\ln \varepsilon| + 4\ln 2 - \lambda_k}\begin{Bmatrix} A_k \\ B_k \end{Bmatrix} \quad (k = 1, 2, \ldots) \tag{6.6}$$

It can be seen that solution of Eq (6.3) does not exist for all small values of ε.

In fact, when $\varepsilon = \varepsilon_k = 16\exp(-\lambda_k)$ the denominator in the second formula of (6.6) vanishes. Then, by virtue of (6.2) $\{\varepsilon_k\}$ is an infinitely small sequence. However, if $\delta(\theta)$ is a trigonometric polynomial of degree m, the function γ is defined for all positive values of ε less than ε_m, and is also a trigonometric polynomial.

We emphasize that the results of an asymptotic analysis must hold when the parameter ε changes continuously in the range $(0, \varepsilon_0)$, where ε_0 is some fixed number. We cannot achieve this situation in the general case if we formulate the condition for complete satisfaction of the solving equation. We recall that in all the previous calculations we only retained terms $O(1)$ and $O(|\ln \varepsilon|)$, and dropped those which approached zero as the parameter ε decreased. As a consequence of this there is no need for an exact solution of the final equation.

We now return to the case (6.4) and we will assume that the function δ is continuously differentiable along Γ, and also has a bounded second derivative.

Note 6.1. If the base of the punch has a "discontinuity" at a certain point (i.e. the first approximation as regards the function δ breaks down), the asymptotic constructions given above lose their meaning for the reason that in the neighbourhood of the discontinuity the stress state of the half-space will be three-dimensional and is, therefore, not described by a plane boundary layer.

On the other hand, the second derivative of δ cannot take greater values because the curvature of the base of the punch must not be too large. Otherwise, we would contradict the assumption that contact must be made over the whole set $\Gamma(\varepsilon)$.

Suppose $|\delta''(\theta)| \leqslant C^{(2)}$, $\theta \in [0, 2\pi)$. Then, for each small $\varepsilon > 0$ the trigonometric polynomial

$$\gamma_\varepsilon(\theta) = \frac{a_0}{2} + \sum_{k=1}^{N_\varepsilon}(a_k \cos k\theta + b_k \sin k\theta), \quad N_\varepsilon = [\varepsilon^{-1}] + 1 \tag{6.7}$$

in which the coefficients are defined by (6.6), while the integer part is denoted by the square brackets, a residual of the order of ε remains in Eq. (6.3). In fact, since

$$\begin{Bmatrix} A_k \\ B_k \end{Bmatrix} = \frac{1}{\pi}\int_0^{2\pi}\delta(\theta)\begin{Bmatrix} \cos k\theta \\ \sin k\theta \end{Bmatrix}d\theta = -\frac{1}{\pi k^2}\int_0^{2\pi}\delta(\theta)''\begin{Bmatrix} \cos k\theta \\ \sin k\theta \end{Bmatrix}d\theta$$

the residual, the remaining γ_ε in (6.3), can be estimated from the relation

$$\left|\pi \sum_{k=N_\varepsilon+1}^{\infty}(A_k \cos k\theta + B_k \sin k\theta)\right| \leqslant \sqrt{2}C^{(2)}\sum_{k=N_\varepsilon+1}^{\infty}\frac{1}{k^2} \leqslant$$

$$\leqslant \sqrt{2}C^{(2)}\left(\sum_{k=2}^{\infty}\frac{1}{k^2-1} - \sum_{k=2}^{N_\varepsilon}\frac{1}{k^2-1}\right) = \sqrt{2}C^{(2)}\frac{2N_\varepsilon+1}{2N_\varepsilon(N_\varepsilon+1)} \leqslant \frac{\sqrt{2}C^{(2)}}{N_\varepsilon}$$

We have used formulae (0.237.3) and (0.133) from [37] above.

Hence, bearing in mind the inequality $1/N_\varepsilon \leqslant \varepsilon$, we conclude that

$$\max_{\theta\in[0,2\pi)}\left|\gamma_\varepsilon(\theta)(|\ln\varepsilon|+4\ln 2)+(J\gamma_\varepsilon)(\theta)-\pi\delta(\theta)\right|\leqslant\sqrt{2}C^{(2)}\varepsilon \tag{6.8}$$

The function (6.7) does not satisfy Eq. (6.3), but because of the smallness of its residual in this equation (see (6.8)) it is suitable for constructing the asymptotic form of the boundary-value problem (1.2).

7. EXAMPLES

1. *A ring-shaped punch with a flat base.* The intensity of the contact pressure, calculated per unit length of the median circle of radius R, mainly satisfies the equation

$$P(\theta)\left(\left|\ln\frac{h}{R}\right|+4\ln 2\right)+(JP)(\theta)=\frac{\pi E}{2(1-\nu^2)}(\delta_0+\beta_2 R\cos\theta-\beta_1 R\sin\theta) \tag{7.1}$$

Here h is the half-width of the ring, δ_0 is a translational variable, and β_1 and β_2 are the angles of rotation of the punch about the y_1 and y_2 axes, respectively.

If the punch is indented without misalignments (the axisymmetrical problem), the force acting on the punch is related to its penetration by the equation

$$Q=\frac{\pi E}{2(1-\nu^2)}\frac{2\pi R\delta_0}{\ln(R/h)+4\ln 2} \tag{7.2}$$

This equation is essentially similar to the leading term of the asymptotic form for Q constructed earlier [5] (compare (7.2) with (0.2)).

By (6.6) the solution of Eq. (7.1) has the form

$$P(\theta)=\frac{\pi E}{2(1-\nu^2)}\left[\frac{\delta_0}{\Lambda}+\frac{R}{\Lambda-2}(\beta_2\cos\theta-\beta_1\sin\theta)\right],\quad \Lambda=\ln\left(16\frac{r}{h}\right)$$

If (as is usually the case) instead of δ_0 and β_1, β_2 we know the value Q of the force pressing the punch, and the coordinates of its point of application (y_1^*, y_2^*), then, in addition to (7.2), we also have from the equilibrium equations of the punch

$$\begin{Bmatrix}\beta_1\\ \beta_2\end{Bmatrix}=\frac{2(1-\nu^2)}{\pi E}\frac{\Lambda-2}{\pi R^3}\begin{Bmatrix}-y_2^*\\ y_1^*\end{Bmatrix}$$

Note that conditions (4.3) will break down if $(y_1^*/R)^2+(y_2^*/R)^2 \geqslant 1/2)^2$. We then encounter the problem of an unknown contact area, which is considerably more complicated than the (linear) problem considered.

The expressions for the forces and moments acting on a non-flat ring-shaped punch are as follows:

$$Q=\frac{\pi E}{2(1-\nu^2)}\frac{R}{\Lambda}\int_0^{2\pi}\delta(\theta)d\theta$$

$$\begin{Bmatrix}M_2\\ -M_1\end{Bmatrix}=\begin{Bmatrix}y_1^*\\ y_2^*\end{Bmatrix}Q=\frac{\pi E}{2(1-\nu^2)}\frac{R^2}{\Lambda-2}\int_0^{2\pi}\delta(\theta)\begin{Bmatrix}\cos\theta\\ \sin\theta\end{Bmatrix}d\theta$$

2. *The effect of a load acting outside the punch.* We can write the vertical displacements of the boundary of the half-space in the Boussinesq problem (see, for example, [27]) at points lying on a circle of radius R (Fig. 1) as

$$-\frac{\pi E}{1-\nu^2}\frac{L}{Q_1}u_3(Q_1;\theta)=\frac{1}{\sqrt{1-2\cos\theta+k^2}}=\frac{A_0}{2}+\sum_{m=1}^{\infty}A_m\cos m\theta \tag{7.3}$$

$$A_0=\pi^{-1}\mathbf{K}(k),\quad A_1=4(\pi k)^{-1}[\mathbf{K}(k)-\mathbf{E}(k)] \tag{7.4}$$

$$A_m=2\left(\frac{k}{2}\right)^{m-2}\sum_{n=1}^{\infty}\frac{(2n-3)!!}{(n-1)!}\cdot\frac{(2n+2m-3)!!}{(n+m-1)!}\left(\frac{k}{2}\right)^{2n}\quad (m=2,3,\ldots) \tag{7.5}$$

Here $k=R/L$, while the distance from the point of application of the point force Q_1 to the centre of the circle will be denoted by L; K and E are the complete elliptic integrals of the first and second kind. Formulae (7.4) correspond to (3.674.1) and (3.674.3) of [31], while (7.5) is obtained from (9.112) of [31].

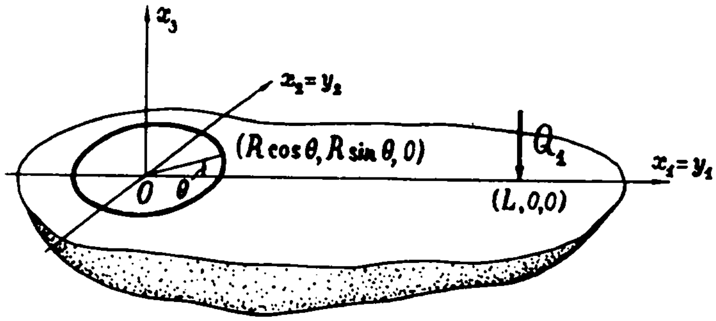

Fig. 1.

If the punch is impressed so that its base can be assumed to be parallel to the $x_3 = 0$ plane, the pressure per unit length should mainly satisfy the equation

$$P(\theta)\Lambda + (\mathbf{J}P)(\theta) = \pi E[2(1-\nu^2)]^{-1}(\delta_0 + u_3(Q_1;\theta)) \tag{7.6}$$

the solution of which is given by (6.5)–(6.7) with the sole difference that, for practical needs, in the sum (6.7) it is sufficient to retain only the first few terms, since the right-hand side (7.6) is infinitely differentiable.

To achieve this gradual penetration of the punch we need to apply the following force to it

$$Q = \frac{1}{\Lambda}\left[\frac{\pi^2 E}{1-\nu^2}R\delta_0 - 2k\mathbf{K}(k)Q_1\right] \tag{7.7}$$

with eccentricity

$$\frac{y_1^*}{R} = -\frac{\Lambda}{\Lambda - 2}\frac{[\mathbf{K}(k) - \mathbf{E}(k)]Q_1}{\pi^2 E[2(1-\nu^2)]^{-1}R\delta_0 - k\mathbf{K}(k)Q_1}$$

If the position of the force which is pressing the punch is fixed (for example, if we assume that its axis of action passes through the centre of the punch), the resultant displacement of the punch will also include a certain rotation. Then, we have the following equation for the density P

$$P(\theta)\Lambda + (\mathbf{J}P)(\theta) = \pi E[2(1-\nu^2)]^{-1}(\delta_0 + \beta_2 R\cos\theta + u_3(Q_1;\theta))$$

the solution of which can also be found from (6.7)–(6.7). Here the value of the settlement of the punch δ_0 is found from (7.7) using the known value of the force Q, while the rotation of the punch β_2 is found from the condition for the principal moment of the loads acting on the punch about the y_2 axis (which intersects the line of action of the force Q) to be zero, and is equal to

$$\beta_2 = \frac{4(1-\nu^2)Q}{\pi^2 ER^2}[\mathbf{K}(k) - \mathbf{E}(k)] \tag{7.8}$$

3. *The logarithmic asymptotic form.* By (5.5) and (5.6) the pressure per unit length under the punch, which we can represent in plan by a narrow curvilinear ring of constant thickness $2h$, the median line of which forms an ellipse with eccentricity e and semimajor axis a, can be found from the equation

$$P(\theta)\left[\Lambda + \frac{1}{2}\ln(1 - e^2\cos^2\theta)\right] + (\mathbf{J}P)(\theta) + (k_1P)(\theta) = \frac{\pi E}{2(1-\nu^2)}\delta(\theta), \quad \Lambda = \ln\frac{16a}{h} \tag{7.9}$$

This equation (like, however, the equations which arose earlier) contains a large parameter, and hence its solution can be expanded in an asymptotic series in inverse powers of Λ

$$P(\theta) - \Lambda^{-1}P_1(\theta) + \Lambda^{-2}P_2(\theta) + \ldots \tag{7.10}$$

$$P_1(\theta) = \pi E[2(1-\nu^2)]^{-i}\delta(\theta) \tag{7.11}$$

$$P_2(\theta) = -\pi E[2(1-\nu^2)]^{-1}\{2^{-1}\delta(\theta)\ln(1-e^2\cos^2\theta)+(\mathbf{J}\delta)(\theta)+(k_1\delta)(\theta)\} \tag{7.12}$$

$$P_{j+1}(\theta) = -2^{-1}P_j(\theta)\ln(1-e^2\cos^2\theta)-(\mathbf{J}P_j)(\theta)-(k_1P_j)(\theta) \quad (j=1,2,\ldots) \tag{7.13}$$

Using the first two terms (7.11) and (7.12) of expansion (7.10) we can obtain qualitatively the correct contact pressure distribution pattern, while for a sufficiently narrow punch we can also obtain numerical results that are quite suitable for practical calculations.

If the eccentricity of the ellipse is small, the right-hand side of (7.12) in turn can be expanded in a power series in e^2. Thus, for a punch with a flat horizontal base

$$P(\theta) = \frac{\pi E}{2(1-\nu^2)}\frac{\delta_0}{\Lambda}\left[1+\frac{e^2(1+\cos^2\theta)}{6\Lambda}+O\left(\frac{e^4}{\Lambda}\right)\right] \tag{7.14}$$

As might have been expected, the maxima of the pressure per unit length (7.14) occur at the points of greatest curvature of the median line.

4. *The interaction between ring-shaped punches.* In the same way as was done in Sections 2–4 (see also [18]), it is also not difficult to derive the solving equations for the pressures per unit length for a system of punches when contact occurs over several narrow regions.

As an example, the system of equations which describes the interaction between two ring-shaped punches with flat bases, takes the form

$$P^{(1)}(\theta)\ln\frac{16R_1}{h_1}+(\mathbf{J}P^{(1)})(\theta) = \frac{\pi E}{2(1-\nu^2)}(\delta_0^{(1)}+\beta_2^{(1)}R_1\cos\theta-\beta_1^{(1)}R_1\sin\theta)-$$

$$-\frac{k_2}{2}\int_{-\pi}^{\pi}\rho(\theta,\tau)^{-1}P^{(2)}(\tau)d\tau \tag{7.15}$$

$$P^{(2)}(\tau)\ln\frac{16R_2}{h_2}+(\mathbf{J}P^{(2)})(\tau) = \frac{\pi E}{2(1-\nu^2)}(\delta_0^{(2)}+\beta_2^{(2)}R_2\cos\tau-\beta_1^{(2)}R_2\sin\tau)-$$

$$-\frac{k_1}{2}\int_{-\pi}^{\pi}\rho(\theta,\tau)^{-1}P^{(1)}(\theta)d\theta \tag{7.16}$$

$$\rho(\theta,\tau)^2 = 1+(k_1-k_2)^2-2(k_1\cos\theta+k_2\cos\tau)+4k_1k_2\cos^2[(\theta+\tau)/2]$$

(the notation used is shown in Fig. 2), where $k_1 = R_1/L$ and $k_2 = R_2/L$.

The approximate solution of the problem for a system of widely separated punches can be constructed using the results obtained in paragraph 2 of this section, which uses ideas proposed previously in [32, Section 7]. Namely, removing one of the punches, we determine the contact pressure under it on the assumption that the action of the remaining punch on the half-space is replaced by the action of a point force applied at the centre of its median circle. Here, by (7.7), the approximate expressions for the forces acting on the punches is found from the system of equations (cyclic substitution of the subscripts 1 and 2)

$$Q_1+\frac{2k_1\mathbf{K}(k_1)}{\ln(16R_1/h_1)}Q_2 = \frac{\pi^2 E}{1-\nu^2}\frac{R_1\delta_0^{(1)}}{\ln(16R_1/h_1)} \quad (1\leftrightarrow 2)$$

If the geometrical characteristics of the punch bases are the same and here $\delta_0^{(1)} = \delta_0^{(2)}$, $\beta_1^{(1)} = \beta_1^{(2)}\beta_2^{(1)} = -\beta_2^{(2)}$, then $P^{(1)} = P^{(2)}$, and Eqs (7.15) and (7.17) reduce to a single equation

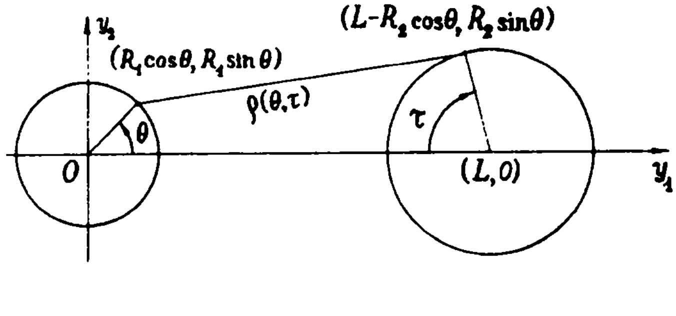

Fig. 2.

$$P(\theta)\ln(16R/h)+(JP)(\theta)+\frac{k}{2}\int_{-\pi}^{\pi}\left\{1-2\left[2k\cos\frac{\theta+\tau}{2}\right]\cos\frac{\theta-\tau}{2}+\left[2k\cos\frac{\theta+\tau}{2}\right]^2\right\}^{-1/2}P(\tau)d\tau=$$

$$=\frac{\pi E}{2(1-\nu^2)}(\delta_0+\beta_2 R\cos\theta-\beta_1 R\sin\theta)$$

In this special case, when we know a priori the forces $Q_1 = Q_2$ impressing the punches, the lines of action of which pass through the centres of the punches, formula (7.8) gives an approximate expression for the value of the angle at which the punches are inclined to one another.

The axisymmetric problem for narrow ring-shaped punches for the most part reduces simply to a system of linear algebraic equations relating the forces acting on the punches to the indentations of the punches.

5. *Solution of the inverse problem*. We will consider the problem of determining the punch thickness for known values of the indentation and the pressure per unit length (compare with [18]).

In particular, if $\delta(\theta) \equiv \delta_0$ and $P(\theta) \equiv P_0$, we have (see Section 5)

$$h(\theta)=16\rho\left|\omega'(\rho e^{i\theta})\right|\exp\left\{-\frac{\pi E}{2(1-\nu^2)}\frac{\delta_0}{P_0}+(k_1 1)(\theta)\right\}$$

$$(k_1 1)(\theta)=\frac{1}{2}\int_{\theta-\pi}^{\theta+\pi}K_1(\theta,\tau)d\tau$$

In general, the solution of this problem can also be reduced to mechanical quadratures.

8. CONCLUDING OBSERVATIONS

First, the process of constructing the asymptotic form can easily be extended; the necessary basis for this is provided by [19–22].

Further, the asymptotic representations of the solutions written in the previous sections, generally speaking, have a formal character and therefore cannot serve as a basis for Galin's hypothesis; the errors must be characterized accurately. Since an even extension of the solution of problem (1.2) from $\mathbf{R}^3$ to $\mathbf{R}^3\backslash\{\mathbf{x} \in \mathbf{R}^3\colon s \in [0, 2l), -\varepsilon H(s) < n < \varepsilon H(s), x_3 = 0\}$ leads to a problem which has already been investigated, we can refer in part to the results obtained in [22–24], where the residuals in the asymptotic formulae were estimated in various metrics. In particular, estimates were given for the deviations in modulus and in the "energy" norm. Due to the presence of singularities (at the punch edges) when considering higher-order derivatives it is necessary to change to weighted norms—the corresponding inequalities can also be found in [33].

The method of matched asymptotic expansions has not been widely used in the theory of contact problems (see the review [3] and also [35]). The form of the punch represented here is not, of course, the only one for which this method is effective. One can, for example, use it to investigate the problem of a punch with a base formed by removing one or several spaced small zones from the flat area. We emphasize that this problem is much simpler than (1.1) or (1.2), since (in accordance with classification [33]) the singular perturbation of the boundary in it is local (concentrated in small neighbourhoods of isolated points), and in (1.2) it is non-local (it is "spread" along a curve).

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